

Equidistribution estimates for eigenfunctions and eigenvalue bounds for random operators

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We discuss properties of L^2 -eigenfunctions of Schrödinger operators and elliptic partial differential operators. The focus is set on unique continuation principles and equidistribution properties. We review recent results and announce new ones.

Keywords: scale-free unique continuation property, equidistribution property, observability estimate, uncertainty relation, Carleman estimate, Schrödinger operator, elliptic differential equation

1. Introduction

In this note we present recent results in Harmonic Analysis for solutions of (time-independent) Schrödinger equations and other partial differential equations. They are motivated by interest in techniques relevant for proving localization for random Schrödinger operators. The mentioned Harmonic Analysis results which we present are a quantitative unique continuation principle and an equidistribution property for eigenfunctions, which is scale-uniform. These results, and variants thereof, go under various names, depending on the particular field of mathematics: They are called observability estimate, uncertainty relation, scale-free unique continuation principle, or local positive definiteness. The latter term signifies that a self-adjoint operator is (strictly) positive definite when restricted to a relevant subspace, while

it is not so on the whole Hilbert space. For the purpose of motivation we discuss this property in the next section.

The term *localization* refers to the phenomenon, that quantum Hamiltonians describing the movement of electrons in certain disordered media exhibit pure point spectrum in appropriately specified energy regions. The corresponding eigenfunctions decay exponentially in space. The (time-dependent) wavepackets describing electrons stay localized essentially in a compact region of space for all times. Note bene, all mentioned properties hold *almost surely*. This is natural in the context of random operators.

An important partial result for deriving localization are Wegner estimates. These are bounds on the expected number of eigenvalues in a bounded energy interval of a random Schrödinger operator restricted to a box.

The localization problem has been studied for other classes of random operators beyond those of Schrödinger type. An example are random divergence type operators, see e.g. Refs. 1 and 2. These are partial differential operators with randomness in coefficients of higher order terms. In particular, the second order term is no longer the Laplacian, but a variable coefficient operator. In this context one is again lead to consider the above mentioned questions of Harmonic Analysis for eigenfunctions of differential operators. In this note we present an exposition of recently published results, and an announcement of a quantitative unique continuation principle and an equidistribution estimate for eigenfunctions for a class of elliptic operators with variable coefficients.

1.1. *Motivation: Moving and lifting of eigenvalues*

Here we discuss some aspects of eigenvalue perturbation theory. It will provide an accessible explanation why one is interested in the results presented in Sections 2 and 3 below in the context of random Schrödinger operators and elliptic differential operators, respectively. In fact, to illustrate the main questions it will be for the moment completely sufficient to restrict our attention to the finite dimensional situation, i.e. to perturbation theory for finite symmetric matrices. The focus will be on how (local) positive definiteness of the perturbation relates to lifting of eigenvalues.

Let A and B be symmetric $n \times n$ matrices, with $B \geq b > 0$ positive definite. The variational min-max principle for eigenvalues shows that for any $k \in \{1, \dots, n\}$ and $t \geq 0$

$$\lambda_k(A + tB) \geq \lambda_k(A) + bt \quad (1)$$

where $\lambda_k(M)$ denotes the k th lowest eigenvalue, counting multiplicities, of a symmetric matrix M . Note that the dimension n does not enter in the bound (1). Without the positive definiteness assumption on B this universal bound will fail, most blatantly if

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \text{Id} & 0 \\ 0 & -\text{Id} \end{pmatrix}.$$

In this case, all eigenvalue $\lambda_k(A + tB)$ *will move*, even with constant speed w.r.t. the variable t , albeit in different directions. If B is singular, some eigenvalues may not move at all. However, for appropriate classes of symmetric matrices A , and of positive semidefinite matrices B , one may still aim to prove

$$\forall t \geq 0, k \in \{1, \dots, n\} \exists \kappa > 0 \text{ such that } \lambda_k(A + tB) \geq \lambda_k(A) + \kappa t \quad (2)$$

Note however, that κ is now not a uniform bound but depends on

- the class of symmetric matrices from which A is chosen,
- the class of semidefinite matrices from which B is chosen,
- the range from which the coupling t is chosen, and
- the range from which the index $k \in \{1, \dots, n\}$ is chosen.

In the case of random operators or matrices one is interested in the situation where

$$A(\omega) = A_0 + \sum_{j \in Q} \omega_j B_j = \left(A_0 + \sum_{j \in Q, j \neq 0} \omega_j B_j \right) + \omega_0 B_0 \quad (3)$$

is a multi-parameter pencil. Here Q is some subset of \mathbb{Z}^d containing 0. The real variables ω_j model random coupling constants determining the strength of the perturbation B_j in each configuration $\omega = (\omega_j)_{j \in Q}$. Now, (3) already suggest to write $A(\omega)$ as

$$A(\omega_0^\perp) + tB \quad \text{where } t = \omega_0, B = B_0, \text{ and } \omega_0^\perp = (\omega_j)_{j \in Q, j \neq 0}.$$

This highlights that if we consider $A(\omega)$ as a function of the single variable $t = \omega_0$, it is clearly a one-parameter family of operators, albeit the “unperturbed part” $A(\omega_0^\perp)$ of $A(\omega) = A(\omega_0^\perp) + tB$ is not a single operator, but varying over the ensemble $(A(\omega_0^\perp))_{\omega_0^\perp}$. To have a useful version of (2) in this situation, the constant κ needs to have a uniform lower bound $\inf_A \kappa$ where $A = A(\omega_0^\perp)$ varies over all matrices in the ensemble.

In what follows we present rigorous results of the type (2), but where A and B are not finite matrices, but differential and multiplication operators. The relevant operators have all compact resolvent, ensuring that the entire spectrum consists of eigenvalues.

2. Equidistribution property of Schrödinger eigenfunctions

The following result is taken from Ref. 3. It is an equidistribution estimate for Schrödinger eigenfunctions, which is uniform w.r.t. the naturally arising length scales, and has strong implications for the spectral theory of random Schrödinger operators.

We fix some notation. For $L > 0$ we denote by $\Lambda_L = (-L/2, L/2)^d$ a cube in \mathbb{R}^d . For $\delta > 0$ the open ball centered at $x \in \mathbb{R}$ with radius δ is denoted by $B(x, \delta)$. For a sequence of points $(x_j)_j$ indexed by $j \in \mathbb{Z}^d$ we denote the collection of balls $\cup_{j \in \mathbb{Z}^d} B(x_j, \delta)$ by S and its intersection with Λ_L by S_L . We will be dealing

with certain subspaces of the standard second order Sobolev space $W^{2,2}(\mathbb{L}_L)$ on the cube. Let Δ be the d -dimensional Laplacian. Its restriction to the cube $\mathbb{L} = \mathbb{L}_L$ needs boundary conditions to be self-adjoint. The domain of the Dirichlet Laplacian will be denoted by $\mathcal{D}(\Delta_{\mathbb{L},0})$ and the domain of the Laplacian with periodic boundary conditions by $\mathcal{D}(\Delta_{\mathbb{L},\text{per}})$. Let $V: \mathbb{R}^d \rightarrow \mathbb{R}$ be a bounded measurable function, and $H_L = (-\Delta + V)_{\Lambda_L}$ a Schrödinger operator on the cube Λ_L with Dirichlet or periodic boundary conditions. The corresponding domains are still $\mathcal{D}(\Delta_{\mathbb{L},0})$ and $\mathcal{D}(\Delta_{\mathbb{L},\text{per}})$, respectively. Note that we denote a multiplication operator by the same symbol as the corresponding function.

The following theorem was proven in Ref. 3.

Theorem 2.1 (Scale-free unique continuation principle). *Let $\delta, K_V > 0$. Then there exists $C_{\text{sfUC}} \in (0, \infty)$ such that for all $L \in 2\mathbb{N} + 1$, all measurable $V: \mathbb{R}^d \rightarrow [-K_V, K_V]$, all real-valued $\psi \in \mathcal{D}(\Delta_{L,0}) \cup \mathcal{D}(\Delta_{L,\text{per}})$ with $(-\Delta + V)\psi = 0$ almost everywhere on Λ_L , and all sequences $(x_j)_{j \in \mathbb{Z}^d} \subset \mathbb{R}^d$, such that for all $j \in \mathbb{Z}^d$ the ball $B(x_j, \delta) \subset \Lambda_1 + j$, we have*

$$\int_{S_L} \psi^2 \geq C_{\text{sfUC}} \int_{\Lambda_L} \psi^2. \quad (4)$$

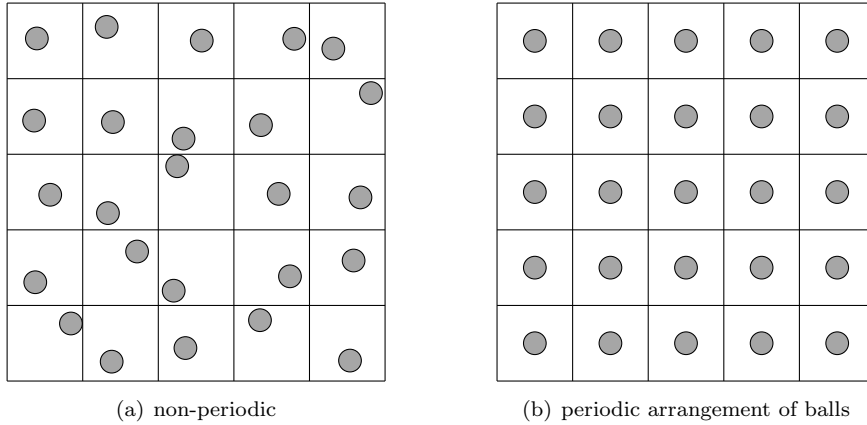


Fig. 1. Examples of collections of balls S_L within region $\Lambda_L \subset \mathbb{R}^2$.

The value of the result is not in the *existence* of the constant C_{sfUC} , but in the *quantitative control* of the dependence of C_{sfUC} on parameters entering the model. The very formulation of the theorem states that C_{sfUC} is independent of the position of the balls $B(x_j, \delta)$ within $\Lambda_1 + j$, and independent of the scale $L \in 2\mathbb{N} + 1$. From the estimates given in Section 2 of Ref. 3 one infers that C_{sfUC} depends on the potential V only through the norm $\|V\|_\infty$ (on an exponential scale), and it depends on the small radius $\delta > 0$ polynomially, i.e. $C \gtrsim \delta^N$, for some $N \in \mathbb{N}$ which depends on the dimension on d and $\|V\|_\infty$.

The theorem states a property of functions in the kernel of the operator. It is easily applied to eigenfunctions corresponding to other eigenvalues since

$$H_L \psi = E \psi \Leftrightarrow (H_L - E) \psi = 0.$$

As a consequence of the energy shift the constant K_V has to be replaced with K_{V-E} , which may be larger than K_V . It may always be estimated by $K_{V-E} \leq K_V + |E|$.

There is a very natural question supported by earlier results, which was spelled out in Ref. 3, namely does the following generalisation of Theorem 2.1 hold: Given $\delta > 0$, $K \geq 0$ and $E \in \mathbb{R}$ there is a constant $C > 0$ such that for all measurable $V: \mathbb{R}^d \rightarrow [-K, K]$, all $L \in 2\mathbb{N} + 1$, and all sequences $(x_j)_{j \in \mathbb{Z}^d} \subset \mathbb{R}^d$ with $B(x_j, \delta) \subset \Lambda_1 + j$ for all $j \in \mathbb{Z}^d$ we have

$$\chi_{(-\infty, E]}(H_L) W_L \chi_{(-\infty, E]}(H_L) \geq C \chi_{(-\infty, E]}(H_L), \quad (5)$$

where $W_L = \chi_{S_L}$ is the indicator function of S_L and $\chi_I(H_L)$ denotes the spectral projector of H_L onto the interval I . Here $C = C_{\delta, K, E}$ is determined by δ, K, E alone.

Klein obtained a positive answer to the question for sufficiently short subintervals of $(-\infty, E]$.

Theorem 2.2 (Ref. 4). *Let $d \in \mathbb{N}$, $E \in \mathbb{R}$, $\delta \in (0, 1/2]$ and $V: \mathbb{R}^d \rightarrow \mathbb{R}$ be measurable and bounded. There is a constant $M_d > 0$ such that if we set*

$$\gamma = \frac{1}{2} \delta^{M_d} (1 + (2\|V\|_\infty + E)^{2/3}),$$

then for all energy intervals $I \subset (-\infty, E]$ with length bounded by 2γ , all $L \in 2\mathbb{N} + 1$, $L \geq 72\sqrt{d}$ and all sequences $(x_j)_{j \in \mathbb{Z}^d} \subset \mathbb{R}^d$ with $B(x_j, \delta) \subset \Lambda_1 + j$ for all $j \in \mathbb{Z}^d$

$$\chi_I(H_L) W_L \chi_I(H_L) \geq \gamma^2 \chi_I(H_L). \quad (6)$$

This does not answer the above posed question completely due to the restriction $|I| \leq 2\gamma$. However, the result is sufficient for many questions in spectral theory of random Schrödinger operators. For a history of the questions discussed here and earlier results we refer to Ref. 3.

2.1. Random Schrödinger operators

Let \mathbb{L}_L be a cube of side $L \in 2\mathbb{N} + 1$, (Ω, \mathbb{P}) a probability space, $V_0: \mathbb{L}_L \rightarrow \mathbb{R}$ a bounded, measurable deterministic potential, $V_\omega: \mathbb{L}_L \rightarrow \mathbb{R}$ a bounded random potential and $H_{\omega, L} = (-\Delta + V_0 + V_\omega)_{\mathbb{L}_L}$ a random Schrödinger operator on $L^2(\mathbb{L}_L)$ with Dirichlet or periodic boundary conditions. We assume that the random potential is of Delone-Anderson form

$$V_\omega(x) := \sum_{j \in \mathbb{Z}^d} \omega_j u_j(x).$$

The random variables $\omega_j, j \in \mathbb{Z}^d$, are independent with probability distributions μ_j , such that for some $m > 0$ and all $j \in \mathbb{Z}^d$ we have $\text{supp } \mu_j \subset [-m, m]$. Fix

$0 < \delta_- < \delta_+ < \infty$ and $0 < C_- \leq C_+ < \infty$. The sequence of measurable functions $u_j: \mathbb{R}^d \rightarrow \mathbb{R}$, $j \in \mathbb{Z}^d$, is such that

$$\forall j \in \mathbb{Z}^d: \quad C_- \chi_{B(z_j, \delta_-)} \leq u_j \leq C_+ \chi_{B(z_j, \delta_+)}, \text{ and } B(z_j, \delta_-) \subset \mathbb{L}_1 + j.$$

2.2. Lifting of eigenvalues

Let $\lambda_k^L(\omega)$ denote the eigenvalues of $H_{\omega, L}$ enumerated in non-decreasing order and counting multiplicities and $\psi_k = \psi_k^L(\omega)$ the normalised eigenvectors corresponding to $\lambda_k^L(\omega)$. While we suppress the dependence of ψ_k on L and ω in the notation, it should be kept in mind. Then

$$\lambda_k^L(\omega) = \langle \psi_k, H_{\omega, L} \psi_k \rangle = \int_{\mathbb{L}_L} \overline{\psi_k} (H_{\omega, L} \psi_k).$$

Define the vector $e = (e_j)_{j \in \mathbb{Z}^d}$ by $e_j = 1$ for $j \in \mathbb{Z}^d$. Consider the monotone shift of V_ω

$$V_{\omega+t \cdot e} = \sum_{j \in \mathbb{Z}^d} (\omega_j + t) u_j$$

and set $Q = Q_L = \Lambda_L \cap \mathbb{Z}^d$. By first order perturbation theory we have

$$\frac{d}{d\tau} \lambda_k^L(\omega + \tau \cdot e)|_{\tau=t} = \langle \psi_k, \sum_{j \in Q} u_j \psi_k \rangle.$$

Note that the right hand side depends on t implicitly through the eigenfunction ψ_k . Let us fix some $E_0 \in \mathbb{R}$ and restrict our attention only to those eigenvalues satisfying $\lambda_n^L(\omega) \leq E_0$. By Theorem 2.1 there exists a constant C_{sFUC} depending on the energy E_0 , δ_- and the overall supremum

$$\sup_{|s| \leq m} \sup_{|\omega_j| \leq m} \sup_{x \in \mathbb{R}^d} |V_0(x) + V_\omega(x) + s \sum_{j \in Q} u_j|$$

of the potential, such that

$$\sum_{k \in Q} \langle \psi_k, u_j \psi_k \rangle \geq C_- \sum_{k \in Q} \langle \psi_k, \chi_{B(z_k, \delta_-)} \psi_k \rangle \geq C_- \cdot C_{\text{sFUC}} =: \kappa.$$

Here we used that $\|\psi\|_{L^2(\Lambda)} = 1$. (Note that the quantity κ depends a-priori on the model parameters.) Integrating the derivative gives

$$\begin{aligned} \lambda_k^L(\omega + t \cdot e) &= \lambda_k^L(\omega) + \int_0^t \frac{d\lambda_k^L(\omega + \tau \cdot e)}{d\tau} \Big|_{\tau=s} ds \\ &\geq \lambda_k^L(\omega) + \int_0^t \kappa ds = \lambda_k^L(\omega) + t\kappa. \end{aligned} \tag{7}$$

This is the lifting estimate for eigenvalues of random (Schrödinger) operators alluded to in §1.1. It should be compared with (2) there. Indeed, due to the uniform nature of the estimate in Theorem 2.1 we have

$$\inf_{L \in 2\mathbb{N}+1} \inf_{\omega \text{ s.t. } \forall k: |\omega_j| \leq m} \inf_{|t| \leq m} \inf_{n \text{ s.t. } \lambda_n^L(\omega) \leq E_0} \kappa > 0. \tag{8}$$

Thus eigenvalues lifting estimate is almost as uniform as (1). A parameter, with respect to which the lifting estimate is *not* uniform is the cut-off energy E_0 . Indeed, if we add in (8) an infimum over $E_0 > 0$ on the left hand side, it becomes zero, unless $\sum_k \chi_{B(z_k, \delta_-)} \geq 1$ almost everywhere on \mathbb{R}^d .

2.3. Wegner estimates

Here we present a Wegner estimate. Such estimates play an important role in the proof of localization via the multiscale analysis. The latter is an induction argument over increasing length scales. The Wegner bound is used to prove the induction step.

Let $s: [0, \infty) \rightarrow [0, 1]$ be the global modulus of continuity of the family $\{\mu_j\}_{j \in \mathbb{Z}^d}$, that is,

$$s(\epsilon) := \sup_{j \in \mathbb{Z}^d} \sup_{a \in \mathbb{R}} \mu_j \left(\left[a - \frac{\epsilon}{2}, a + \frac{\epsilon}{2} \right] \right)$$

The main result of Ref. 3 on the model described in the last paragraph is a Wegner estimate which is valid for all compact energy intervals.

Theorem 2.3 (Ref. 3). *Let $H_{\omega, L}$ be a random Schrödinger operator as in §2.1. Then for each $E_0 \in \mathbb{R}$ there exists a constant C_W , such that for all $E \leq E_0$, $\epsilon \leq 1/3$, and all $L \in 2\mathbb{N} + 1$ we have*

$$\mathbb{E}\{\text{Tr} [\chi_{[E-\epsilon, E+\epsilon]}(H_{\omega, L})]\} \leq C_W s(\epsilon) |\ln \epsilon|^d |\Lambda_L|.$$

The Wegner constant C_W depends only on E_0 , $\|V_0\|_\infty$, m , C_- , C_+ , δ_- , and δ_+ . Klein⁴ obtains an improvement over this result based on his above quoted Theorem 2.2. There are many earlier, related Wegner estimates. For an overview we refer to Ref. 3.

2.4. Comparison of local L^2 -norms

An important step in the proof of Theorem 2.1 is the following result which compares L^2 -norms of the restrictions of a PDE-solution to two distinct subsets. In our applications the solution will be an eigenfunction of the Schrödinger operator. Various estimates of this type have been given in Refs. 5, 6 and 3. We quote here the version from the last mentioned paper.

Theorem 2.4. *Let $K, R, \beta \in [0, \infty)$, $\delta \in (0, 1]$. There exists a constant $C_{\text{qUC}} = C_{\text{qUC}}(d, K, R, \delta, \beta) > 0$ such that, for any $G \subset \mathbb{R}^d$ open, any $\Theta \subset G$ measurable, satisfying the geometric conditions*

$$\text{diam } \Theta + \text{dist}(0, \Theta) \leq 2R \leq 2 \text{dist}(0, \Theta), \quad \delta < 4R, \quad B(0, 14R) \subset G,$$

and any measurable $V: G \rightarrow [-K, K]$ and real-valued $\psi \in W^{2,2}(G)$ satisfying the differential inequality

$$|\Delta \psi| \leq |V \psi| \quad \text{a.e. on } G \quad \text{as well as} \quad \int_G |\psi|^2 \leq \beta \int_\Theta |\psi|^2,$$

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we have

$$\int_{B(0,\delta)} |\psi|^2 \geq C_{\text{qUC}} \int_{\Theta} |\psi|^2. \quad (9)$$

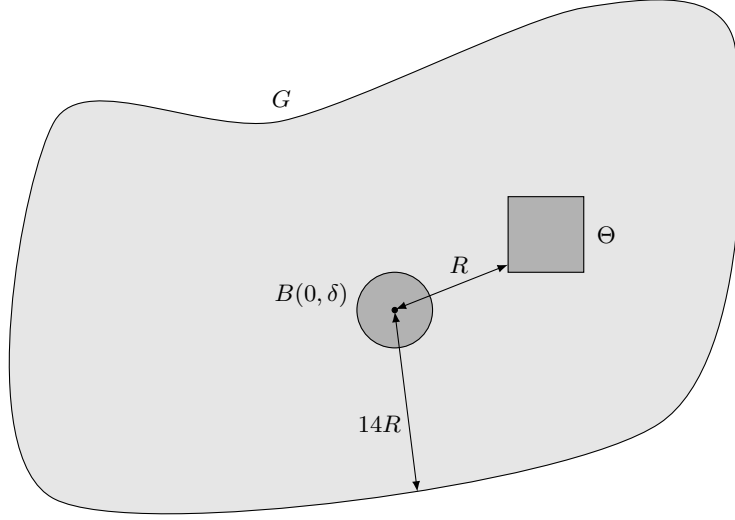


Fig. 2. Assumptions in Theorem 2.4 on the geometric constellation of G , Θ , and $B(0, \delta)$

3. Equidistribution property eigenfunctions of second order elliptic operators

3.1. Notation

Let \mathcal{L} be the second order partial differential operator

$$\mathcal{L}u = - \sum_{i,j=1}^d \partial_i (a^{ij} \partial_j u)$$

acting on functions u on \mathbb{R}^d . Here ∂_i denotes the i th weak derivative. Moreover, we introduce the following assumption on the coefficient functions a^{ij} .

Assumption 3.1. *Let $r, \vartheta_1, \vartheta_2 > 0$. The operator \mathcal{L} satisfies $A(r, \vartheta_1, \vartheta_2)$, if and only if $a^{ij} = a^{ji}$ for all $i, j \in \{1, \dots, d\}$ and for almost all $x, y \in B(0, r)$ and all $\xi \in \mathbb{R}^d$ we have*

$$\vartheta_1^{-1} |\xi|^2 \leq \sum_{i,j=1}^d a^{ij}(x) \xi_i \xi_j \leq \vartheta_1 |\xi|^2 \quad \text{and} \quad \sum_{i,j=1}^d |a^{ij}(x) - a^{ij}(y)| \leq \vartheta_2 |x - y|.$$

3.2. A quantitative unique continuation principle

We first present an extension of the quantitative continuation principle, formulated for Schrödinger operators in Theorem 2.4, to elliptic operators with variable coefficients.

Theorem 3.1 (Ref. 7). *Let $R \in (0, \infty)$, $K_V, \beta \in [0, \infty)$ and $\delta \in (0, 4R]$. There is an $\epsilon > 0$, such that if $A(14R, 1 + \epsilon, \epsilon)$ holds then there is a constant $C_{\text{qUC}} > 0$, such that for any open $G \subset \mathbb{R}^d$ containing the origin and $\Theta \subset G$ measurable satisfying*

$$\text{diam } \Theta + \text{dist}(0, \Theta) \leq 2R \leq 2 \text{dist}(0, \Theta) \quad \text{and} \quad B(0, 14R) \subset G,$$

any measurable $V : G \rightarrow [-K_V, K_V]$ and real-valued $\psi \in W^{2,2}(G)$ satisfying the differential inequality

$$|\mathcal{L}\psi| \leq |V\psi| \quad \text{a.e. on } G \quad \text{as well as} \quad \frac{\|\psi\|_G^2}{\|\psi\|_\Theta^2} \leq \beta,$$

we have

$$\|\psi\|_{B(x,\delta)}^2 \geq C_{\text{qUC}} \|\psi\|_\Theta^2. \quad (10)$$

3.3. Scale-free unique continuation principle

We move on to discuss the equidistribution property or scale-free unique continuation principle for eigenfunctions. The aim is to formulate an analog of Theorem 2.1 for variable coefficient elliptic operators. As presented below, for the moment we have solved only the situation where the second order term is sufficiently close to the Laplacian.

As before, we denote by L_L a box of side $L \in \mathbb{N}$. By V we indicate a bounded measurable potential on \mathbb{R}^d taking values in $[-K_V, K_V]$, where K_V is a positive constant. We restrict the operator \mathcal{L} on $L_L(0)$ and add either periodic or Dirichlet boundary conditions. In the former case we denote such an operator by $\mathcal{L}_{L,0}$, and its domain $\mathcal{D}(\mathcal{L}_{L,0})$ is the subspace of $W^{2,2}(L_L)$ consisting of functions vanishing on ∂L_L . The notation for the operator with periodic boundary condition is $\mathcal{L}_{L,\text{per}}$ and its domains $\mathcal{D}(\mathcal{L}_{L,\text{per}})$ consists of the functions in $W^{2,2}(L_L)$ satisfying periodic boundary conditions.

Assumption 3.2. *For each pair i, j the function $a^{ij} : \mathbb{R}^d \rightarrow \mathbb{R}$ is \mathbb{Z}^d -periodic.*

Assume that in the case of operator $\mathcal{L}_{L,0}$ its coefficients a^{ij} , $i \neq j$ vanish on the sides of box L_L , while the coefficients a^{ii} satisfy periodic boundary conditions on the sides of box L_L . In the case of operator $\mathcal{L}_{L,\text{per}}$ suppose that all its coefficients satisfy periodic boundary conditions on the sides of box L_L .

Theorem 3.2. *Fix $K_V \in [0, +\infty)$, $\delta \in (0, 1]$. Assume $A(\sqrt{d}, 1 + \epsilon, \epsilon)$ with $\epsilon > 0$ as in Theorem 3.1. Assume 3.2.*

Then there exists a constant $C_{\text{sfUC}} > 0$ such that for any $L \in 2\mathbb{N} + 1$, any sequence

$$Z := \{z_k\}_{k \in \mathbb{Z}^d} \quad \text{in } \mathbb{R}^d \quad \text{such that } B(z_k, \delta) \subset L_1(k) \text{ for each } k \in \mathbb{Z}^d,$$

any measurable $V : L_L \mapsto [-K_V, K_V]$ and any real-valued $\psi \in \mathcal{D}(\mathcal{L}_{L,0})$, respectively $\psi \in \mathcal{D}(\mathcal{L}_{L,\text{per}})$ satisfying

$$|\mathcal{L}\psi| \leq |V\psi| \quad \text{a.e.} \quad L_L$$

we have

$$\int_{S_L} |\psi(x)|^2 dx = \sum_{k \in Q_L} \|\psi\|_{L^2(B(z_k, \delta))}^2 \geq C_{sfUC} \|\psi\|_{L^2(L_L)}^2, \quad (11)$$

where $S_L := S \cap L_L = \cup_{k \in Q_L} B(z_k, \delta)$, $Q_L = L_L \cap \mathbb{Z}^d$, and $S := \cup_{k \in \mathbb{Z}^d} B(z_k, \delta)$.

As a *Corollary* we obtain immediately an eigenvalue lifting estimate analogous to (7), where κ is again uniform w.r.t. many parameters, as spelled out in subsection 2.2 explicitly.

The proof of Theorem 3.2 is based on the strategy implemented in Ref. 3. First one uses the conditions on the coefficients a^{ij} described in Assumption 3.2 to extend ψ as well as the differential expression \mathcal{L} to the whole of \mathbb{R}^d while keeping the $W^{2,2}$ -regularity and the differential inequality originally satisfied by ψ . Then one uses the comparison Theorem 3.1 for local L^2 -norms. Note that now the condition concerning the minimal distance to the boundary of G plays no role, since ψ has been extended to the whole of \mathbb{R}^d . From this point the combinatorial and geometric arguments of Ref 3 take over. In fact, one can prove a abstract meta-theorem: Once the comparison of local L^2 -norms of ψ holds up to the boundary, an equidistribution property for ψ follows. Interestingly, such an argument no longer uses the fact that ψ is a solution of an differential equation or inequality.

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